End-Semester Exam : Random Walks on Graphs.

Exam Date : May 13, 2021.

Submit solutions via Moodle by 1.30 PM on May 13th.

Please write and sign the following declaration on your answer script first :

I have not received, I have not given, nor will I give or receive, any assistance to another student taking this exam, including discussing the exam with other students. The solution to the problems are my own and I have not copied it from anywhere else. I have used only class notes and the reference textbooks prescribed for the course.

Attempt any four questions only. Each question carries 10 points. If you attempt more than four questions, only the first four answers will be evaluated.

1. Let (G,μ) be a weighted locally finite connected graph

   (a) Assume that G is *spherically symmetric* i.e., there exists a $o \in G$ such that for all $x, y \in G$ with $d(x,o) = d(y,o)$, there exists an automorphism $\phi : (G,\mu) \to (G,\mu)$ satisfying $\phi(o) = o, \phi(x) = y$. Here an automorphism of weighted graphs means that $\phi : G \to G$ is a graph automorphism such that if e is incident on x then so is $\phi(e)$ on $\phi(x)$ and $c(e) = c(\phi(e))$ for all $x \in V, e \in E$. Define

   $$C_n = \sum_{(x,y)\in E} c(x,y)\mathbf{1}[d(o,x) = n-1, d(o,y) = n].$$

   Show that the simple random walk on (G,μ) is transient iff $\sum_{n\geq 1} C_n^{-1} < \infty$.

   (b) Let $f, g \in \mathbb{H}^2$ and set $h = f \wedge g$. Let $U = \{x \in V : f(x) > g(x)\}$. Assume that $U \subset B(o,N)^c$ for some $N \geq 1$. Show that there exists a constant c independent of N such that

   $$\mathcal{E}(f-h, f-h) \leq c\left(\mathcal{E}_{B(o,N-1)^c}(f) + \mathcal{E}(g)\right).$$

2. (a) Let (G,μ) be a recurrent graph. Let $o \in V, n \geq 1$, and $B(o,n)$ be the ball of radius n around o. Define $h_n : V \to [0,1]$ by

   $$h_n(x) = \mathbb{P}_x(\tau_{B^c(o,n)} < \tau_o).$$

   Show that h_n is harmonic on $B(o,n) \setminus \{o\}$ and $\lim_{n\to\infty} h_n(x) = 0$ for all $x \in V$.

   (b) If the RW on (G,μ) is transient show that $\psi(z) = \mathbb{P}_z(\tau_x^+ < \infty)$ is a non-constant super-harmonic function.

3. Consider the graph $G = (V,E)$ formed by joining two copies of $\mathbb{Z}^3$ at the origin. Show that G does not satisfy the Liouville Property.

4. Let (G,μ) be a weighted locally finite connected graph satisfying the Elliptic Harnack inequality. Show the following.

   (a) Let $K > 1$ and $R \geq \frac{2}{K-1}$. Show that for any non-negative function h defined on $\bar{B}(o,KR)$ and harmonic on $B(o,KR)$, there exists a constant $C_H(K)$ such that

   $$h(x) \leq C_H(K)h(y), \ \forall x, y \in B(o,R).$$

(b) Let K, R be as above. Set $D = B(o, KR)$ and τ_{D^c} be the hitting time of D^c and let $B \subset \partial D$. Show that there exists a constant C such that for all $x, y \in B(o, R)$, we have that

$$\mathbb{P}_x(X_{\tau_{D^c}} \in B) \leq C\mathbb{P}_y(X_{\tau_{D^c}} \in B),$$

where X_n is the SRW on (G, μ).

5. Let p and $\tilde{p}$ be two increment probability measures of a reversible and irreducible stochastic matrices on a finite group G. Let S and $\tilde{S}$ be the respective symmetric generating sets. Let $P_a := \{(s_1, \dots, s_k) : a = s_1 \dots s_k, s_i \in S\}$ be all the possible ways to write a in terms of elements of S and ν_a, a probability measure on P_a and $N(s, \gamma)$ be the number of times s occurs in $\gamma \in P_a$. Show that $\tilde{\gamma}_2 \leq B\gamma_2$ where

$$B := \max_{s \in S} \left(\frac{1}{p(s)} \sum_{a \in \tilde{S}} \tilde{p}(a) \sum_{\gamma \in P_a} \nu_a(\gamma) N(s, \gamma) |\gamma| \right).$$

6. Give a complete proof that $(F^2_\infty), (N_\infty)$ and (S^2_∞) are equivalent on locally finite, connected, weighted graphs. The approximation arguments are to be justified.

7. (a) Let (G, μ) be a a weighted locally finite connected graph. Let f_1, f_2 be non-negative functions with compact supports $A_1, A_2 \subset V$ respectively such that $R = d(A_1, A_2) := \min\{d(x, y) : x \in A_1, y \in A_2\}$. Show that

$$\langle P^n f_1, f_2 \rangle \leq 2\|f_1\|_2\|f_2\|_2 e^{-R^2/2n}.$$

(b) Suppose that G_1, G_2 are weighted graphs and $G = G_1 \otimes G_2$ is their tensor product. Suppose that G_1, G_2 satisfy Nash inequality for α_1, α_2 with $\alpha_i \geq 1, i = 1, 2$. Show that G satisfies Nash inequality with $\alpha = \alpha_1 + \alpha_2$.

8. Give a complete proof of stability under rough isometry of the weak Poincaré inequality. Please justify all the inequalities.